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CONGRUENCE THEOREMS FOR RIEMANNIAN SUBMANIFOLDS

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ABSTRACT

The aim of this paper is, essentially, to give sufficient conditions in terms of mean curvature for two submanifolds of a given Riemannian manifold to be congruent modulo a given 1-parameter group of transformations. The results obtained generalise those of several authors including M. Okumura and the present author [11].

1. Introduction

We first recall some results which relate to the following general problem. Let $(M,\ g)$ be a smooth Riemannian manifold of dimension $r+1$, let $\phi:\ \mathbf{R}\times M\rightarrow M$ be a 1-parameter group of transformations of M generated by a vector field ξ, and let $W,\ \bar{W}$ be smooth, compact, oriented r-dimensional connected hypersurfaces in M such that the points of $\bar{W}$ are given by the formula $\phi(f(q),\ q),\ q\in W$, where $f:W\rightarrow\mathbb{R}$ is a given smooth function. For each $q\in W$ define a hypersurface W_q by the condition that the map $\phi_{f(q)}:W\rightarrow W_q;\ \ x\rightarrow\phi(f(q),\ x)$ should be a diffeomorphism. Also for each $q\in W$ write $\bar{q}=\phi(f(q),\ q)$ and define

$$\bar{H}(\bar{q})=\ \text{mean curvature of }\bar{W}\text{ at }\bar{q},$$
$$H(\bar{q})=\ \text{mean curvature of }W_q\text{ at }\bar{q},$$

where the above maps are assumed to preserve orientation. Then one may ask: Under what further conditions does the equality $\bar{H}(\bar{q})=H(\bar{q})$ for all $q\in W$ imply that W and $\bar{W}$ are congruent modulo ϕ, that is $\bar{W}=\phi_c(W)$ for some $c\in\mathbb{R}$?

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Two cases of particular interest arise as follows: Suppose W and $\bar{W}$ are surfaces in Euclidean space E^3 and $\theta : W \to \bar{W}$ is a smooth map preserving orientation. Also, suppose the set of points $p \in W$ where the lines $[p, \bar{p}]$, $\bar{p} = \theta(p)$, are tangent to W does not have inner points. Then the following two theorems apply.

THEOREM 1.1: (H. Hopf and K. Voss [9], K. Voss [12], P. Hartman [4]) *If all the lines $[p, \bar{p}]$ are parallel and θ preserves the mean curvature of W and $\bar{W}$ (that is $H(p) = \bar{H}(\bar{p})$), then $\bar{W}$ is obtained from W by a translation of E^3.*

THEOREM 1.2: (A. Aeppli ([1]) *If all the lines $[p, \bar{p}]$ pass through a fixed point 0 and if $r\tilde{H}(p) = \bar{r}\bar{\tilde{H}}(\bar{p})$, where r and $\bar{r}$ are the distances from 0 to p and $\bar{p}$ respectively, and $\tilde{H}(p)$ is the mean curvature of W at p, then $\bar{W}$ is obtained from W by a homothety, thus the ratio $\bar{r}/r$ is constant.*

In the above two theorems ϕ is a 1-parameter group of translations or homotheties.

Next, with the notation of the first paragraph, let S be the set of singular points of $\bar{W}$, that is the set of points on which the vector field ξ is tangential to $\bar{W}$. Then the following three theorems are known:

THEOREM 1.3: (Y. Katsurada [10]) *If $\bar{H}(\bar{q}) = H(\bar{q})$ for all $q \in W$, and ϕ is a 1-parameter group of homothetic transformations for which S is nowhere dense in $\bar{W}$, then W and $\bar{W}$ are congruent modulo ϕ.*

THEOREM 1.4: (H. Hopf and Y. Katsurada [8]) *If $\bar{H}(\bar{q}) = H(\bar{q})$ for all $q \in W$ and the set S is empty, then W and $\bar{W}$ are congruent modulo ϕ.*

The method of proof of Theorem 1.4 is to show that f satisfies an elliptic partial differential equation on W and then to apply the well-known maximum principle of E. Hopf [7]. By modifying the maximum principle, using a special case of a theorem due to P. Hartman and R. Sacksteder [5], H. Brühlmann has generalised Theorem 1.4 as follows:

THEOREM 1.5: (H. Brühlmann [3]) *Define $G : \bar{W} \to \mathbb{R}$ by $G = g(\bar{N}, \xi)$ where $\bar{N}$ is the unit normal vector field over $\bar{W}$. If* grad *$G \neq 0$ whenever $G = 0$ on $\bar{W}$, and $\bar{H}(\bar{p}) = H(\bar{p})$ for all $p \in W$, then the hypersurfaces W and $\bar{W}$ are congruent modulo ϕ.*

Our main purpose here is to generalise Theorem 1.4 in the following ways. Firstly, we remove the conditions of orientability, then we replace the compactness conditions on W and W by the condition that f should attain a maximum value on W; next, we allow W to have arbitrary codimension and remove the assumption that $\bar{W}$ is a submanifold of M; finally we replace the global condition that $\bar{H} = H$ by a local condition on mean curvature vector fields. Indeed, our proof is essentially a local one and does not require the method of variation of mean curvature found in [8]. We then obtain a similar generalisation of Brühlmann's theorem by considering the above conditions in greater detail.

2. Extension of the Hopf–Katsurada theorem

The following notation and assumptions will apply from hereon. Let $(M,\ g)$ be a smooth Riemannian manifold of dimension $r+n$, let $\phi : \mathbb{R} \times M \to M$ be a 1-parameter group of transformations of M generated by a vector field ξ, and let W be a smooth connected r-dimensional submanifold of M. For each $t \in \mathbb{R}$ define the map $\phi_t :\ W \to M$ by $\phi_t(x) = \phi(t,\ x)$. We consider a given smooth function $f : W \to \mathbb{R}$ and write $\bar{W} = \psi(W)$ where the map $\psi : W \to M$ is defined by

$$\psi(q) = \phi(f(q),\ q) \quad \text{for all } q \in W.$$

As before we write $\psi(q)$ as $\bar{q}$. However, we remark that $\bar{W}$ is not assumed to have a submanifold structure. Now assume f attains a maximum value, say c, on W. If $f(p) = c$ then $df = 0$ at p so $d\psi = d\phi_c$ at p where $d\psi$ and $d\phi_c$ denote the differentials of ψ and ϕ_c acting on $T_p(W)$, the tangent space to W at p. It follows that ψ is an immersion of some neighbourhood of p in W into M. Hence there exists a neighbourhood V of p in W and an embedded submanifold $\bar{V}$ of M such that $\bar{V} = \Psi(V)$ and the map $V \to \bar{V};\ \ x \mapsto \Psi(x)$ is a diffeomorphism. Similarly, for each $q \in V$ define a submanifold $V_{\bar{q}}$ of M such that $V_{\bar{q}} = \phi_{f(q)}(V)$ and the map $V \to V_{\bar{q}};\ \ x \mapsto \phi_{f(q)}(x) = \phi(f(q),\ x)$ is a diffeomorphism. We note that $\bar{q} \in V_{\bar{q}}$ for each $\bar{q} \in \bar{V}$, also the tangent spaces $T_{\bar{q}}(\bar{V})$ and $T_{\bar{p}}(V_{\bar{p}})$ agree. Then we define vector fields $\bar{H}$ and H along $\bar{V}$ by writing $\bar{H}(\bar{q})$ and $H(\bar{q})$ for the mean curvature vectors of $\bar{V}$ and $V_{\bar{q}}$ at $\bar{q}$ for each $\bar{q} \in \bar{V}$. Also, we define a smooth function $F :\ \bar{V} \to \mathbb{R}$ by $f|V = F \circ \psi$. Finally, define an equivalence relation $\sim$ on $C^\infty(\bar{V})$ by writing $F_1 \sim F_2$ if $F_1 - F_2 = X(F)$ for some smooth vector field X tangential to $\bar{V}$. We now prove the following theorem.

THEOREM 2.1: *With the above notation, suppose f attains on W a maximum value, say c. For each maximum point p of f choose V and $\bar{V}$ as above and suppose there exists a smooth unit normal vector field $\bar{N}$ on $\bar{V}$ such that*

(i) $g(H,\ \bar{N}) \sim g(\bar{H},\ \bar{N})$

and

(ii) $g(\bar{N},\ \xi) \neq 0$ at $\bar{p}$.

Then $\bar{W}$ is a submanifold of M and W, $\bar{W}$ are congruent modulo ϕ.

Proof: We use the above notation relating to a maximum p of f. In particular we choose V and $\bar{V} = \psi(V)$ as above and remark that any further restrictions of V or $\bar{V}$ are understood to be applied simultaneously so as to preserve the relation $\psi(V) = \bar{V}$. Now restrict V so as to obtain a chart* $V\{v^\alpha\}$ at p on W, then a chart $\bar{V}\{\bar{v}^\alpha\}$ is defined by the relation $v^\alpha = \bar{v}^\alpha \circ \phi$. By restricting $\bar{V}$ we may assume a chart $U\{x^i\}$ is defined at $\bar{p}$ on M such that $\bar{V} \subset U$. We write g_{ij} and g^{ij} for the covariant and contravariant components of the metric tensor field g on $U\{x^i\}$ and $\bar{h}_{\alpha\beta}$ and $\bar{h}^{\alpha\beta}$ for the covariant and contravariant components of the induced metric tensor field $\bar{h}$ on $\bar{V}\{\bar{v}^\alpha\}$. Next, we note that since f and ϕ are continuous, V can be restricted so that $\phi_{f(q)}(V) \subset U$ for all $q \in V$. Then for each $\bar{q} \in \bar{V}$ a chart $V_{\bar{q}}\{v^\alpha_{\bar{q}}\}$ is defined on the submanifold $V_{\bar{q}}$ by the relation

$$v^\alpha = v^\alpha_{\bar{q}} \circ \phi_{f(q)}. \tag{1}$$

In order to consider the vector fields $\bar{H}$ and H described above, we first use the relation $v^\alpha = v^{-\alpha} \circ \psi$ to obtain

$$\begin{aligned}\frac{\partial x^i}{\partial \bar{v}^\alpha} \circ \psi &= \frac{\partial (x^i \circ \psi)}{\partial v^\alpha} \\ &= \frac{\partial (x^i \circ \psi)}{\partial v^\alpha} + \frac{\partial (x^i \circ \psi)}{\partial t}\frac{\partial f}{\partial v^\alpha}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 x^i}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} \circ \psi &= \frac{\partial^2 (x^i \circ \psi)}{\partial v^\alpha \partial v^\beta} \\ &= \frac{\partial^2 (x^i \circ \phi)}{\partial v^\alpha \partial v^\beta} + \frac{\partial^2 (x^i \circ \phi)}{\partial v^\alpha \partial t}\frac{\partial f}{\partial v^\beta} + \frac{\partial^2 (x^i \circ \phi)}{\partial v^\beta \partial t}\frac{\partial f}{\partial v^\alpha} \\ &\quad + \frac{\partial^2 (x^i \circ \phi)}{\partial t^2}\frac{\partial f}{\partial v^\alpha}\frac{\partial f}{\partial v^\beta} + \frac{\partial (x^i \circ \phi)}{\partial t}\frac{\partial^2 f}{\partial v^\alpha \partial v^\beta},\end{aligned}$$

the right hand sides being evaluated where $t = f(q)$ for each $q \in V$. For the corresponding equations on $\bar{V}$ we use the function F and the vector field ξ, as defined above, to obtain

$$\frac{\partial x^i}{\partial \bar{v}^\alpha} = \frac{\partial (x^i \circ \phi)}{\partial v^\alpha} \circ \psi^{-1} + \xi^i \frac{\partial F}{\partial \bar{v}^\alpha} \tag{2}$$

---

* Greek suffixes indicate the range $1, \ldots, r$ and Roman suffixes indicate the range $1, \ldots, r+n$. A repeated suffix indicates summation.

and

$$
\begin{aligned}
\frac{\alpha^2 x^i}{\partial\bar{v}^\alpha \partial\bar{v}^\beta} =& \frac{\partial^2(x^i \circ \phi)}{\partial v^\alpha \partial v^\beta} \circ \psi^{-1} + \frac{\partial \xi^i}{\partial x^j}\left[\frac{\partial x^j}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} + \frac{\partial x^j}{\partial \bar{v}^\beta}\frac{\partial F}{\partial \bar{v}^\alpha}\right] \\
& - \frac{\partial \xi^i}{\partial x^j}\xi^j \frac{\partial F}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} + \xi^i \frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}
\end{aligned} \tag{3}
$$

at all points of $\bar{V}$, where $\xi = \xi^i \partial/\partial x^i$ on U.

We note from (1) that if each x^i is restricted to $V_{\bar{q}}$ then

$$
\begin{cases}
\frac{\partial x^i}{\partial v^\alpha_{\bar{q}}}(\bar{q}) = \frac{\partial (x^i \circ \phi)}{\partial v^\alpha}(\psi^{-1}(\bar{q})), \\
\frac{\partial^2 x^i}{\partial v^\alpha_{\bar{q}} \partial v^\beta_{\bar{q}}}(\bar{q}) = \frac{\partial^2 (x^i \circ \phi)}{\partial v^\alpha \partial v^\beta}(\psi^{-1}(\bar{q})).
\end{cases} \tag{4}
$$

Now for each $\bar{q} \in \bar{V}$ the natural components $h_{\alpha\beta}(\bar{q})$ of the induced metric on $V_{\bar{q}}$ at $\bar{q}$ are given by

$$
h_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial v^\alpha_{\bar{q}}}\frac{\partial x^j}{\partial v^\beta_{\bar{q}}}.
$$

Hence, from (2) and (4), the functions $h_{\alpha\beta}$ on $\bar{V}$ are given by

$$
\begin{aligned}
h_{\alpha\beta} &= g_{ij}\left[\frac{\partial x^i}{\partial \bar{v}^\alpha} - \xi^i \frac{\partial F}{\partial \bar{v}^\alpha}\right]\left[\frac{\partial x^j}{\partial \bar{v}^\beta} - \xi^j \frac{\partial F}{\partial \bar{v}^\beta}\right] \\
&= \bar{h}_{\alpha\beta} + g(\xi,\, \xi)\frac{\partial F}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} - g(e_\alpha,\, \xi)\frac{\partial F}{\partial \bar{v}^\beta} - g(e_\beta,\, \xi)\frac{\partial F}{\partial \bar{v}^\alpha}
\end{aligned} \tag{5}
$$

where, from hereon, we write e_α for the vector field $(\partial x^i/\partial \bar{v}^\alpha)\partial/\partial x^i$ defined on $\bar{V}$. Then from (5).

$$
\begin{aligned}
\bar{h}^{\gamma\delta} - h^{\gamma\delta} &= (h_{\alpha\beta} - \bar{h}_{\alpha\beta})h^{\alpha\gamma}\bar{h}^{\beta\delta} \\
&= \left[g(\xi,\, \xi)\frac{\partial F}{\partial \bar{v}^\alpha}\delta^\epsilon_\beta - g(e_\alpha,\, \xi)\delta^\epsilon_\beta - g(e_\beta,\, \xi)\delta^\epsilon_\alpha\right] h^{\alpha\gamma}\bar{h}^{\beta\delta}\frac{\partial F}{\partial \bar{v}^\epsilon}
\end{aligned} \tag{6}
$$

where $(h^{\gamma\delta})$ denotes the inverse of the matrix $(h_{\gamma\delta})$. It follows that, under the above definition of equivalence,

$$
\bar{h}^{\alpha\beta} \sim h^{\alpha\beta} \tag{7}
$$

for all α, β.

We now use the vector field $\bar{N}$, as given in the theorem, and the mean curvature vector field $\bar{H}$ to obtain

$$
g(\bar{H},\, \bar{N}) = g_{ij}\bar{h}^{\alpha\beta}\bar{N}^i\left[\frac{\partial^2 x^j}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + \Gamma^j_{k\ell}\frac{\partial x^k}{\partial \bar{v}^\alpha}\frac{\partial x^\ell}{\partial \bar{v}^\beta}\right]
$$

where

$$\bar{N} = \bar{N}^i \frac{\partial}{\partial x^i} \quad \text{on } \bar{V} \quad \text{and} \quad \Gamma^j_{kl} \frac{\partial}{\partial x^j} = \frac{\Delta_\partial}{\partial x^k} \frac{\partial}{\partial x^\ell} \quad \text{on } U.$$

Next we consider $g(H, \bar{N})$ and first define a vector field N on $\bar{V}$ by the condition that, for each $\bar{q} \in \bar{V}$, $N_{\bar{q}}$ is on the component of $\bar{N}_{\bar{q}}$ tangential to $V_{\bar{q}}$. Clearly, from (2) and (4)

$$N = N^i \frac{\partial}{\partial x^i} = a^\alpha \left[\frac{\partial x^i}{\partial \bar{v}^\alpha} - \xi^i \frac{\partial F}{\partial \bar{v}^\alpha}\right] \frac{\partial}{\partial x^i} = a^\alpha \left[e_\alpha - \xi \frac{\partial F}{\partial \bar{v}^\alpha}\right] \tag{8}$$

for some functions a^α on $\bar{V}$ such that

$$g\left[\bar{N} - a^\alpha \left[e_\alpha - \xi \frac{\partial F}{\partial \bar{v}^\alpha}\right],\ e_\beta - \xi \frac{\partial F}{\partial \bar{v}^\beta}\right] = 0,$$

thus

(9)

$$a^\alpha \left[h_{\alpha\beta} - g(e_\alpha,\ \xi)\frac{\partial F}{\partial \bar{v}^\beta} - g(e_\beta,\ \xi)\frac{\partial F}{\partial \bar{v}^\alpha} + g(\xi,\ \xi)\frac{\partial F}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta}\right] = -g(\bar{N},\ \xi)\frac{\partial F}{\partial \bar{v}^\beta}.$$

Since $\partial F/\partial \bar{v}^\alpha = 0$ at $\bar{p}$ and $(h_{\alpha\beta})$ is non-singular, it follows from (9) that $\bar{V}$ can be restricted so that $a^\alpha \sim 0$ on $\bar{V}$. Hence $N^i \sim 0$ on $\bar{V}$. Also, from (2), (3) and (4), at each $\bar{q} \in \bar{V}$,

(10)

$$\begin{aligned} g(H,\ \bar{N}) &= g(H,\ \bar{N} - N) \\ &= g_{ij} h^{\alpha\beta}(\bar{N}^i - N^i)\left[\frac{\partial^2 x^j}{\partial v^\alpha_{\bar{q}} \partial v^\beta_{\bar{q}}} + \Gamma^j_{k\ell}\frac{\partial x^k}{\partial v^\alpha_{\bar{q}}}\frac{\partial x^\ell}{\partial v^\beta_{\bar{q}}}\right] \\ &= g_{ij} h^{\alpha\beta}(\bar{N}^i - N^i)\left[\frac{\partial^2 x^j}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} - 2\frac{\partial \xi^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} + \xi^k \frac{\partial \xi^j}{\partial x^k}\frac{\partial F}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta}\right. \\ &\quad \left. - \xi^j \frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + \Gamma^j_{k\ell}\left[\frac{\partial x^k}{\partial \bar{v}^\alpha} - \xi^k \frac{\partial F}{\partial \bar{v}^\alpha}\right]\left[\frac{\partial x^\ell}{\partial \bar{v}^\beta} - \xi^\ell \frac{\partial F}{\partial \bar{v}^\beta}\right]\right]. \end{aligned}$$

Since $N^i \sim 0$ and $h^{\alpha\beta} \sim \bar{h}^{\alpha\beta}$, it follows that

$$\begin{aligned} g(H,\ \bar{N}) &\sim g_{ij}\bar{h}^{\alpha\beta}\bar{N}^i\left[\frac{\partial^2 x^j}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} - 2\frac{\partial \xi^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} + \xi^k \frac{\partial \xi^j}{\partial x^k}\frac{\partial F}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta}\right. \\ &\quad \left. - \xi^j \frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + \Gamma^j_{k\ell}\left[\frac{\partial x^k}{\partial \bar{v}^\alpha} - \xi^k \frac{\partial F}{\partial \bar{v}^\alpha}\right]\left[\frac{\partial x^\ell}{\partial \bar{v}^\beta} - \xi^\ell \frac{\partial F}{\partial \bar{v}^\beta}\right]\right] \\ &\sim g(\bar{H},\ \bar{N}) - g(\bar{N},\ \xi)\ h^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \alpha \bar{v}^\beta}. \end{aligned} \tag{11}$$

Now by hypothesis, $g(\bar{N},\ \xi) \neq 0$ at $\bar{p}$ and we may then restrict $\bar{V}$ so that $g(\bar{N},\ \xi) \neq 0$ on $\bar{V}$. Again, by hypothesis, $g(H,\ \bar{N}) \sim g(\bar{H},\ \bar{N})$ so it follows from (11) that on $\bar{V}$

$$\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} \sim 0.$$

Hence, by Hopf's maximum principle [7], F is locally constant at $\bar{p}$. This implies f is locally constant at each point $p \in W$ at which $f(p) = c$. Since W is assumed to be connected, it follows from the continuity of f that $f = c$ on W. Thus $\bar{W} = \phi_c(W)$ which shows that $\bar{W}$ is a submanifold of M and W, $\bar{W}$ are congruent modulo ϕ as required. ∎

We note from the above proof that the vector field $\bar{N} - N$ on $\bar{V}$ is smooth and takes the value $\bar{N}_{\bar{p}}$ at $\bar{p}$. Thus by restricting $\bar{V}$ we may assume that $\bar{N} - N$ is nowhere zero on $\bar{V}$ and we then write $\tilde{N} = (\bar{N}-N)/\|\bar{N}-N\|$. Thus $\tilde{N}$ is smooth on $\bar{V}$ and, for each $\bar{q} \in \bar{V}$, $\tilde{N}$ is a unit normal to $\bar{V}_{\bar{q}}$. Also $\tilde{N} = \bar{N}$ at $\bar{p}$. We now prove:

COROLLARY 2.2: *Suppose (ii) of Theorem 2.1 is replaced by the condition that on $\bar{V}$*

(ii)′ $g(H,\ \tilde{N}) \sim g(\bar{H},\ \tilde{N})$.

Then, again, $\bar{W}$ is a submanifold of M and W, $\bar{W}$ are congruent modulo ϕ.

Proof: As already shown, the components N^i of the vector field N on $\bar{V}$ satisfy $N^i \sim 0$. Hence

$$\|\bar{N} - N\|^2 = g_{ij}(\bar{N}^i - N^i)(\bar{N}^j - N^j) \sim 1.$$

Then

$$\|\bar{N} - N\| = \frac{\|\bar{N} - N\|^2 - 1}{\|\bar{N} - N\| + 1} + 1 \sim 1$$

and from (ii)′

$$g(H,\ \bar{N}) \sim g(H,\ \bar{N} - N) = \|\bar{N} - N\| g(\bar{H},\ \tilde{N}) \sim g(\bar{H},\ \tilde{N}).$$

The corollary then follows from Theorem 2.1. Also, Theorem 1.4 is now a special case of Corollary 2.2 where W has codimension one in M.

3. Extension of Brühlmann's theorem

We recall from [3] that Brühlmann's proof of Theorem 1.5 depends on a generalisation of Hopf's maximum principle which we now state in a modified form suitable for later use.

LEMMA 3.1: *Let P, $p^{\alpha\beta}$ and q^α, for $\alpha,\beta = 1,\ldots,r$, be smooth real-valued functions on an open ball $D\{u^\alpha\} \subset \mathbf{R}^r$ where the quadratic form $p^{\alpha\beta}\lambda_\alpha\lambda_\beta$ is positive definite, and let Q satisfy the differential equation*

$$Pp^{\alpha\beta}\frac{\partial^2 Q}{\partial u^\alpha \partial u^\beta} + q^\alpha \frac{\partial Q}{\partial u^\alpha} = 0$$

on D. Suppose there exists $u_0 \in D$ such that $Q(u) \leq Q(u_0)$ everywhere on D and suppose either $P(u_0) \neq 0$ or $q^\alpha \partial P/\partial u^\alpha > 0$ at u_0. Then Q is constant on some neighbourhood of u_0.

We now generalise Brühlmann's theorem by modifying the conditions for Theorem 2.1. In particular, we use a new equivalence relation on $C^\infty(\bar{V})$ by writing $F_1 \sim F_2$ if $F_1 - F_2 = X(F)$ for some smooth vector field X on $\bar{V}$ such that $X = 0$ at $\bar{p}$. Then with the notation of §2 we prove

THEOREM 3.2: *Suppose f attains on W a maximum value, say c. For each maximum point p of f choose coordinate neighbourhoods V of p and $\bar{V}$ of $\bar{p}$ as above and suppose there exists a smooth unit normal vector field $\bar{N}$ on $\bar{V}$ such that*

(i) $g(H,\ \bar{N}) \sim g(\bar{H},\ \bar{N})$ and

(ii) *if $g(bN,\ \xi) = 0$ at $\bar{p}$ then $\bar{h}^{\alpha\beta} g(\bar{N},\ \nabla_{e_\alpha}\eta) e_\beta(g(\bar{N},\ \eta)) > 0$ at $\bar{p}$, where η is the normal component of ξ along $\bar{V}$.*

Then $\bar{W}$ is a submanifold of M and W, $\bar{W}$ are congruent modulo ϕ.

Proof: The case where $g(\bar{N},\ \xi) \neq 0$ at $\bar{p}$ has already been considered, so we may assume $g(\bar{N},\ \xi) = 0$ at $\bar{p}$. Then, as a consequence of (8) and (9), $a^\alpha \sim 0$ and $N^i \sim 0$. Also, it follows from (6) that

$$h^{\gamma\delta} - \bar{h}^{\gamma\delta} \sim \left[g(e_\alpha,\ \xi)\frac{\partial F}{\partial \bar{v}^\beta} + g(e_\beta,\ \xi)\frac{\partial F}{\partial \bar{v}^\alpha}\right] \bar{h}^{\alpha\gamma}\bar{h}^{\beta\delta}.$$

Then from (10) and the above equivalences

$$\begin{aligned}
g(H,\ \bar{N}) \sim g_{ij}N^i & \left[\bar{h}^{\alpha\beta} + \bar{h}^{\alpha\gamma}\bar{h}^{\beta\delta}\left[g(e_\gamma,\ \xi)\frac{\partial F}{\partial \bar{v}^\delta} + g(e_\delta,\xi)\frac{\partial F}{\partial \bar{v}^\gamma}\right]\right] \\
& \times \left[\frac{\partial^2 x^j}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + \Gamma^j_{k\ell}\frac{\partial x^k}{\partial \bar{v}^\alpha}\frac{\partial x^\ell}{\partial \bar{v}^\beta}\right. \\
& \left. -2\frac{\partial \xi^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{v}^\alpha}\frac{\partial F}{\partial \bar{v}^\beta} - 2\Gamma^j_{k\ell}\xi^k\frac{\partial x^\ell}{\partial \bar{v}^\beta}\frac{\partial F}{\partial \bar{v}^\alpha} - \xi^j\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}\right] \\
& \sim g(\bar{H},\ \bar{N}) - 2g(\bar{N},\ \nabla_{e_\alpha}\xi)h^{\alpha\beta}\frac{\partial F}{\partial \bar{v}^\beta} - g(\bar{N},\ \xi)\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} \\
& \quad + 2\bar{h}^{\alpha\gamma}\bar{h}^{\beta\delta}g(e_\gamma,\ \xi)\frac{\partial F}{\partial \bar{v}^\delta}g(\bar{N},\ \nabla_{e_\beta}e_\alpha) \\
& = g(\bar{H},\ \bar{N}) - 2g(\bar{N},\ \nabla_{e_\alpha}\xi)h^{\alpha\beta}\frac{\partial F}{\partial \bar{v}^\beta} - g(\bar{N},\ \xi)\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} \\
& \quad - 2g(\nabla_{e_\alpha}\bar{N},\ \xi - \eta)\bar{h}^{\alpha\beta}\frac{\partial F}{\partial \bar{v}^\beta} \\
& = g(\bar{H},\ \bar{N}) - 2g(\bar{N},\ \nabla_{e_\alpha}\eta)\bar{h}^{\alpha\beta}\frac{\partial F}{\partial \bar{v}^\beta} - g(\bar{N},\xi)\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}
\end{aligned}$$

where η denotes the normal component of ξ along $\bar{V}$. Hence, from (i) of the theorem,

$$g(N,\ \xi)\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + 2g(\bar{N},\ \nabla_{e_\alpha}\eta)\bar{h}^{\alpha\beta}\frac{\partial F}{\partial \bar{v}^\beta} \sim 0.$$

Thus, on $\bar{V}$ the function F satisfies a differential equation of the form

$$g(N,\ \xi)\bar{h}^{\alpha\beta}\frac{\partial^2 F}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} + H^\beta\frac{\partial F}{\partial \bar{v}^\beta} = 0$$

and at $\bar{p}$

$$H^\alpha e_\alpha(g(N,\ \xi)) = 2\bar{h}^{\alpha\beta}g(\bar{N},\ \nabla_{e_\alpha}\eta)e_\beta(g(\bar{N},\ \eta)).$$

Consequently, from Lemma 3.1 and the hypothesis of the theorem, F is locally constant at $\bar{p}$. Then, as before, $f = c$ on W and $\bar{W} = \psi_c(W)$, as required. ∎

We note that if W has codimension one in M then $\nabla_{e_\alpha}\bar{N}$ is tangential to $\bar{V}$ so (ii) of Theorem 3.2 can be replaced by the simpler condition that $e_\alpha(g(\bar{N},\ \xi)) \neq 0$. Also, since $N^i \sim 0$, then, as in the proof of Corollary 2.2, we have $g(H,\ \tilde{N}) \sim g(H,\ \bar{N})$. Hence Theorem 1.5 follows as a special case.

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